Theory and implementation of event-triggered stabilization over digital channels

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***Abstract*— In the context of event-triggered control, the timing of the triggering events carries information about the state of the system that can be used for stabilization. At each triggering event, not only can information be transmitted by the message content (data payload) but also by its timing. We demonstrate this in the context of stabilization of a laboratory-scale inverted pendulum around its equilibrium point over a digital communication channel with bounded unknown delay. Our event-triggering control strategy encodes timing information by transmitting in a state-dependent fashion and can achieve stabilization using a data payload transmission rate smaller than what the data-rate theorem prescribes for classical periodic control policies that do not exploit timing information. Through experimental results, we show that as the delay in the communication channel increases, a higher data payload transmission rate is required to fulfill the proposed event-triggering policy requirements. This confirms the theoretical intuition that a larger delay brings a larger uncertainty about the value of the state at the controller, as less timing information is carried in the communication. Our results also provide a novel encoding-decoding scheme to achieve input-to-state practically stability (ISpS) for nonlinear continuous-time systems under appropriate assumptions.**

I. Introduction

Event-triggered control has gained significant attention due to its advantages over conventional control schemes in cyber-physical systems. Although periodic control is the most common and perhaps simplest solution for digital systems, it can be inefficient in sharing communication and computation resources [1], [2]. The central concept of event-triggered control is to transmit sensory data only when needed to satisfy the control objective. In addition to utilizing the distributed resources efficiently, it has been proven that the timing of the triggering events, effectively revealing the state of the system, carries information that can be used for stabilization. This allows achieving stabilization with a transmission rate over the feedback loop smaller than that required by periodic control strategies [3]–[5].

In networked control systems a finite-rate digital communication channel closes the loop between the sensor and the controller. In this setting, data-rate theorems [6]–[8] provide the communication channel requirements for stabilization. They state that to ensure stabilization of an unstable linear system, the minimum information rate communicated over the channel, including both data payload

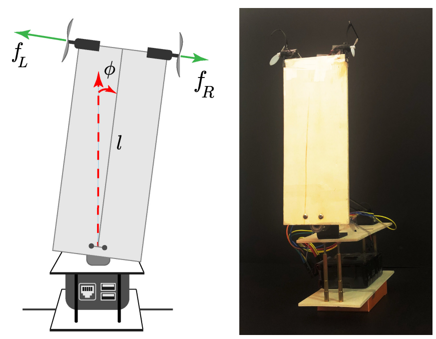

Fig. 1. An inverted pendulum controlled by thrust force of two propellers. The pendulum is a plywood sheet of length l. The angle ϕ of the pendulum from the vertical line and its rate of change, measured by the sensor and transmitted to the controller over a digital channel with bounded unknown delay, are used to determine the left and right thrust forces f_L and f_R of the propellers.

and timing information, must be at least equal to the *entropy rate* of the plant, defined as the sum of the unstable modes in nats [3], [4]. When information is encoded in the timing of the transmission events using event-triggered control, our previous work [9], [10] has shown the existence of an event-triggering strategy that achieves input-to-state practically stability (ISpS) [11], [12] for any linear, time-invariant system subject to bounded disturbance over a digital communication channel with bounded delay using a data payload transmission rate lower than the entropy rate. This is possible because, for small values of the delay, the timing information is substantial, and the data payload transmission rate can be lower than the entropy rate of the plant. However, as the delay increases, a higher data payload transmission rate is required to satisfy the requirements of the proposed event-triggering control strategy.

A similar data-rate theorem formulation also holds for nonlinear systems. The works [13]–[15] for nonlinear systems are restricted to plants without disturbances and with a bit-pipe communication channel. The work [13] uses the entropy of topological dynamical systems to elegantly determine necessary and sufficient bit rates for local uniform asymptotic stability. Consequently, the results are only local and derived under restrictive assumptions. Under appropriate assumptions, the work [14] extends to nonlinear but locally Lipschitz systems, the zoom-in/zoom-out strategy of [16]. The sufficient condition proposed in this work is, however, conservative, and does not match the necessary condition proposed in [13]. The work [12] further extend the results in [14] to linear systems with uncertainty and under appropriate assumptions to nonlinear systems with

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disturbances. Inspired by the Jordan block decomposition employed in [7] to design an encoder/decoder pair of a vector system, the work [15] provides a sufficient design for feed-forward dynamics that matches the necessary condition proposed in [13].

The majority of results on control under communication constraints are restricted to theoretical works. Here for the first time, we examine data-rate theorems in a practical setting, using an inverted pendulum, a classic example of an inherently unstable nonlinear plant with numerous practical applications. Our first contribution is to implement the event-triggering control design introduced in [9], [10], and demonstrate the utilization of timing information to stabilize a laboratory-scale inverted pendulum over a digital communication channel with bounded unknown delay, see Fig. 1. A video that illustrates the main ideas and demonstrates our experimental results can be found at https://youtu.be/1P0i-tWsPoA. The results of our experiments show that using the sufficient packet size derived in [9], [10] on a linearized model of the inverted pendulum around its unstable equilibrium point, the state estimation error is sufficiently small and we can stabilize the system. We show that for small values of the delay the experimental data payload transmission rate is lower than the entropy rate of the plant. On the other hand, by increasing the upper bound on the delay in the communication channel, higher data payload transmission rates are required to satisfy the requirements of the proposed control strategy. The event-triggering policy developed in [9], [10] can only stabilize the pendulum *locally* around its equilibrium point, where linearization is possible. Our second contribution is to address nonlinear systems directly, and develop a novel event-triggering scheme that exploits timing information to render a class of continuous-time nonlinear systems subject to disturbances ISpS.

From the system's perspective, our set-up is closest to the one in [12], [14], as we consider locally Lipschitz nonlinear systems that can be made input-to-state stable (ISS) with respect to the state estimation error and system disturbances. Using our encoding-decoding scheme, we encode the information in timing via event-triggering control in a state-dependent fashion to achieve input-to-state practical stability (ISpS) in the presence of unknown but bounded delay. We also discuss the different approaches to eliminate the ISS assumption.

Finally, we point out that the work [17] studies event-triggering stabilization of globally Lipschitz nonlinear system without disturbances where the communication delay is arbitrarily small. Also, the work [18] investigates event-triggered stabilization of nonlinear system under communication constraints but it does not explicitly quantify the effect of quantization in the presence of system disturbances, nor the timing information carried by the triggering events.

A complete list of notations and proofs of all the results appear in the online appendix [19], due to lack of space.

II. System model

We consider the stabilization of the inverted pendulum depicted in Fig. 1 around its unstable equilibrium point. The sensory information for stabilization is sent to the controller over a digital channel. The block diagram of the control system is given in Fig. 2. We assume the communication

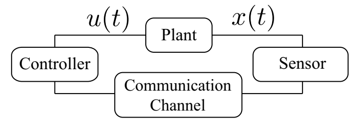

Fig. 2. System model.

channel is capable of transmitting packets composed of a finite number of bits without error. Each transmitted packet is subject an unknown delay upper bounded by $\gamma \geq 0$. In addition to the data payload, the transmission time of the packets sent over the channel could be utilized to convey information to the controller. As a result, the encoding process consists of choosing the timing and data payload of the packet, as shown in Fig. 3. In other words, in the sensor block, the quantized version of the state is encoded in a packet containing data payload as well as its timing. In

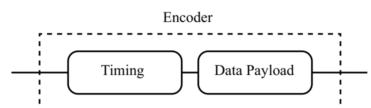

Fig. 3. Representation of information transmission using data payload and transmission time of the packet in a digital channel. The encoding process consists of choosing the data payloads and their transmission times. Here, the sensor determines the transmission time using our event-triggering strategy in a state-dependent manner.

our design, the sensor encodes information in timing via an event-triggering technique in a state-dependent fashion.

At each triggering event, occurring at times $\{t_s^k\}_{k\in\mathbb{N}}$, the sensor transmits a packet $p(t_s^k)$ of length $g(t_s^k)$ over the communication channel. Packets arrive at the controller at times $\{t_c^k\}_{k\in\mathbb{N}}$. When referring to a generic triggering or reception time, we skip the super-script k in t_s^k and t_c^k.

Since the delay in the communication channel is upper bounded by $\gamma \geq 0$, the *communication delays* represented by $\Delta_k = t_c^k - t_s^k$ ($k \in \mathbb{N}$) must satisfy

$$\Delta_k \leq \gamma. \tag{1}$$

By defining the k^{th} *triggering interval* as $\Delta'_k = t_s^{k+1} - t_s^k$, the *information transmission rate* (the rate at which sensor transmits data payload over the channel) can be defined as

$$R_s = \limsup_{N\to\infty} \left(\sum_{k=1}^{N} g(t_s^k) \Big/ \sum_{k=1}^{N} \Delta'_k \right). \tag{2}$$

A. Plant Dynamics

We consider a linearized version of the two-dimensional problem of balancing an inverted pendulum with two propellers, where the motion of the pendulum is constrained in a plane and its position can be measured by an angle ϕ representing small deviations from the upright position of the pendulum, as depicted in Fig. 1. The inverted pendulum has mass m_1 and length l. The propellers are identical and are attached to two motors of mass m_2. m and I respectively represent the total mass of the system and its moment of inertia. Therefore, a nonlinear equation of the system can be written as follows

$$I\ddot{\phi} = mgl\sin\phi(t) + \xi(t)l + \text{noise}, \tag{3}$$

where g is the gravitational acceleration, and $\xi(t)$ is the resultant thrust force of the propellers (f_L and f_R as shown in Fig. 1) generating a moment about the axis of rotation of the pendulum. Note that in this nonlinear equation the effect of the friction is included in the additive noise. The force $\xi(t)$ can be estimated as a linear function of the control input $\tilde{u}(t)$, applied to the motors, with some proportionality constant k_ξ (found from experiments), namely $\xi(t) = k_\xi \tilde{u}(t)$.

We derive the linearized equations of motion using a small angle approximation. This linearization is only valid for sufficiently small values of the delay upper bound γ in the communication channel. Linearizing (3) around the equilibrium point results in the following dynamics

$$I\ddot{\phi} = mgl\phi(t) + k_\xi l\tilde{u}(t) + \text{noise}.$$

By defining the state variable $\tilde{\boldsymbol{x}} = (\phi, \dot{\phi})^T$, the state-space equations can be written as follows

$$\dot{\tilde{\boldsymbol{x}}} = \tilde{\mathbf{A}}\tilde{\boldsymbol{x}} + \tilde{\mathbf{B}}\tilde{u}(t) + \tilde{\boldsymbol{w}}(t), \tag{4}$$

where

$$\tilde{\mathbf{A}} = \begin{bmatrix} 0 & 1 \\ \frac{mgl}{I} & 0 \end{bmatrix}, \tilde{\mathbf{B}} = \begin{bmatrix} 0 \\ \frac{k_\xi l}{I} \end{bmatrix}.$$

In our prototype shown in Fig. 1, the pendulum is a plywood sheet of size $0.18 \times 0.073 \times 0.005$ m and mass $m_1 = 0.030$ kg. The motors are of mass $m_2 = 0.010$ kg. Also, $l = 0.180$ m, and $g = 9.81$ m/s^2. Using first principles, one can find the moment of inertia of the pendulum about its axis of rotation to be $I = 3.57 \times 10^{-4}$ kg/m^2. By experiments, we approximate $k_\xi = 0.001$. Therefore, the system (4) can be rewritten as follows

$$\dot{\tilde{\boldsymbol{x}}} = \begin{bmatrix} 0 & 1 \\ 53.58 & 0 \end{bmatrix}\tilde{\boldsymbol{x}} + \begin{bmatrix} 0 \\ 0.50 \end{bmatrix}\tilde{u}(t) + \tilde{\boldsymbol{w}}(t). \tag{5}$$

Using (4) it follows $\tilde{w}_1(t) = 0$. Also, by experiments we deduce $|w_2(t)|$ is upper bounded by 0.02.

Now using the eigenvector matrix

$$\mathbf{P} = \begin{bmatrix} 0.1354 & -0.1354 \\ 0.9908 & 0.9908 \end{bmatrix}$$

of matrix $\tilde{\mathbf{A}}$ we consider a canonical transformation to diagonalize the system (5) as follows

$$\dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x}(t) + \mathbf{B}u(t) + \boldsymbol{w}(t), \tag{6}$$

where $\mathbf{A} = \mathbf{P}^{-1}\tilde{\mathbf{A}}\mathbf{P}$, $\mathbf{B} = \mathbf{P}^{-1}\tilde{\mathbf{B}}$, $\boldsymbol{x}(t) = \mathbf{P}^{-1}\tilde{\boldsymbol{x}}(t)$ and $\boldsymbol{w}(t) = \mathbf{P}^{-1}\tilde{\boldsymbol{w}}(t)$. Therefore, for the diagonalized system (6) we have

$$\mathbf{A} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 7.3198 & 0 \\ 0 & -7.3198 \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} 0.2523 \\ 0.2523 \end{bmatrix}, \ \boldsymbol{x} = \begin{bmatrix} 3.6940\phi + 0.5046\dot{\phi} \\ 0.5046\dot{\phi} - 3.6940\phi \end{bmatrix},$$

$$|w_i(t)| \leq M = 0.0470 \ \text{ for } i \in \{1, 2\},$$

where the upper bound M on the $|w_i(t)|$ for $i \in \{1, 2\}$ is found by taking the maximum of upper bounds of all the elements in $\boldsymbol{w}(t)$.

We now define a modified version of input-to-state practically stablity (ISpS) [11], [12], which is suitable for our event-triggering setup with the unknown but bounded delay in the digital communication channel.

Definition 1: The plant (6) is ISpS if both of its coordinates $x_1(0)$ and $x_2(0)$ are ISpS. Also, $x_1(t)$ is ISpS if there exist $\beta \in \mathcal{KL}$, $\psi \in \mathcal{K}_\infty(0)$, $d \in \mathbb{R}_{\geq 0}$, $\chi \in \mathcal{K}_\infty(d)$, $d' \in \mathbb{R}_{\geq 0}$ and $\zeta \in \mathcal{K}^2_\infty(0, d')$ such that for all $t \geq 0$

$$|x_1(t)| \leq \beta(|x_1(0)|, t) + \psi(|w_1|_t) + \chi(\gamma) + \zeta(|w_1|_t, \gamma).$$

Note that, for a fixed γ, this definition reduces to the standard notion of ISpS. Given that the initial condition, delay, and system disturbances are bounded, ISpS implies that the state must be bounded at all times beyond a fixed horizon.

Since λ_2 in (6) is negative, the second coordinate is inherently stable, and we do not need to transmit updates about the second coordinate to the controller via the communication channel. However, since λ_1 is positive, the uncertainty about the first coordinate grows exponentially at the controller, hence the sensor needs to communicate information to the controller about the state of the first coordinate to render the plant ISpS [9].

A brief description of the event-triggered control approach in our previous work [9], [10] which determines the sequence of transmission times $\{t^k_s\}_{k\in\mathbb{N}}$ and packets $\{p(t^k_s)\}_{k\in\mathbb{N}}$ to achieves ISpS for the first coordinate of the dynamics (6) is available at App. B [19].

III. Implementation and System Architecture

We now present the details of the implementation of the proposed event-triggered control scheme on a real system, along with experimental results validating the theory. The prototype used is an inverted pendulum system built using off-the-shelf components. The body of the system is made of plywood sheets, as depicted in Fig. 1. For sensors, we use InvenSense MPU6050 MEMS sensor which has a 3-axis accelerometer and a 3-axis gyroscope, and we use a complementary filter to read the angle and angular velocity of the pendulum. We choose Raspberry Pi Model 3 for the computation unit and the controller in the system. For actuation, we use two small DC motors equipped with two identical propellers. Fig. 4 depicts the different components of the system.

Using the plant dynamics introduced in (6), we implement the event-triggered control scheme proposed in App. B on the prototype system. While our theory is developed for continuous-time plants, the experiments are performed on digital systems and in discrete-time domain with small enough sampling time δ to make the discrete-time model as close to the continuous-time model as possible. Because of this discretization, the minimum upper bound for the channel delay is equal to two sampling times. A delay of at most one sampling time exists from the time that a triggering occurs to the time that the sensor takes a sample from the plant state and another delay of at most one sampling time exists from the time that the packet is received to the time the control input is applied to the plant. In the experiments, a

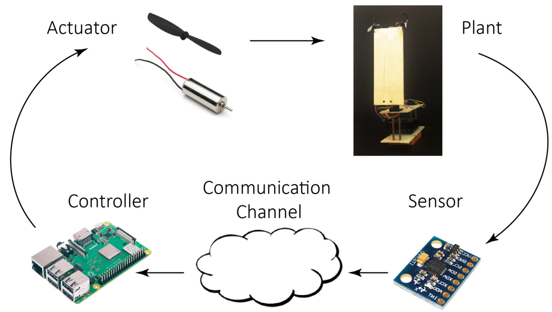

Fig. 4. Architecture and components of the prototype.

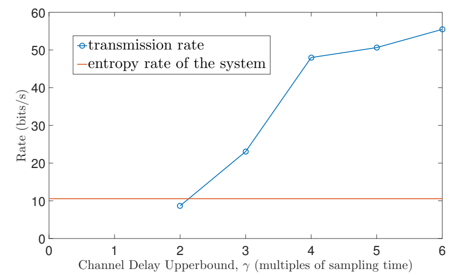

Fig. 5. Information transmission rate in experiments compared with the entropy rate of the system. Note that the rate calculated from experiments does not start at zero worst-case delay because the minimum channel delay upper bound is equal to two sampling times (0.006 seconds). The entropy rate of the system is equal to $\lambda_1/\ln 2 = 10.56$ bits/sec while the minimum transmission rate for worst-case delay equal to two sampling time in the experiments is equal to 8.66 bits/sec.

triggering occurs as soon as z_1 is equal or greater than J and the controller has received the previous packet, in this way since the sampling time is small, at the triggering time, equation (31) will be valid approximately.

To simulate the digital channel between the sensor and the controller, we send packets composed of a finite number of bits from the sensor to the controller with a delay, that is a multiple of the sampling time δ, upper bounded by γ.

IV. Experimental results

In this section, experimental results for various scenarios are presented. In all the experiments, the sampling time δ is 0.003 seconds, which is the smallest sampling time that the measurements from our sensors permit. Also we set $\rho_0 = 0.01$, $b = 1.00001$, and $J = \frac{M}{\lambda_1 \rho_0}(e^{\lambda_1 \gamma} - 1) + 0.1$. In the first set of experiments, we evaluate the performance of the controller for different values of γ. In Fig. 6, the first row presents the results when $\gamma = 0.006$ seconds or two sampling times and the second row presents the results when $\gamma = 0.015$ seconds or five sampling times. The first column is the evolution of the absolute value of the state estimation error (30) (red) in time along with the triggering threshold (blue). As the absolute value of this error is greater than or equal to the triggering function, a triggering occurs and the sensor transmits a packet to the controller. However, due to the random delay (upper bounded by γ) in the communication channel, this error could grow beyond the triggering function. This growth, of course, can become larger as γ increases which is shown in the first column of Fig. 6. The first column also shows, more triggering occurred when the channel delay is upper bounded with five sampling times.

The second column in Fig. 6 presents the evolution of the state x_1 (blue) corresponding to the unstable pole in the diagonalized system (6) and its estimate $\hat{x}_1$ (red) in time. The last column shows the evolution of the actual states of the system, namely the angle of the pendulum in radians and its rate of change in radians/sec. It can be seen that $|\phi|$ remains less than 0.2 radians which ensures the linearization of (3) remains valid and is a good approximation.

We repeat the experiments for different values of γ and calculate the sufficient transmission rate using (2). According to the data-rate theorem, to stabilize the plant, the information rate communicated over the channel in data payload and timing should be larger than the entropy rate of the plant [3], [4]. In our experiments, when $\gamma = 2\delta$ the timing information is substantial, therefore, the information transmission rate becomes smaller than the entropy rate of the plant which is shown in Fig. 5. Furthermore, according to the theory developed in [9], [10] as γ increases, more information has to be sent via data payload for stabilization since larger delay corresponds to more uncertainties about the value of the states at the controller and less timing information.

Remark 1: Similar to our analysis in [9], we assume the plant disturbance is random but bounded. In most of our experiments, we successfully stabilized inverted pendulum around its equilibrium point. Disturbances outside the prescribed limits occur rarely, but can still happen occasionally. Assuming that the disturbances are unbounded one might be able to extend the second-moment stability results of [20] to our setup. Similarly, the case where the delay in the communication channel becomes unbounded with a positive probability is another interesting research problem. •

V. Extension to nonlinear systems

The results developed in [9], [10] are restricted to linear systems, and they can only stabilize the pendulum (3) locally, where the linear approximation is valid. Thus, now we develop a novel event-triggering scheme that encodes information in timing and under appropriate assumptions renders a continuous-time nonlinear system with disturbances ISpS. Clearly, the results of this section compare to the results of [9], [10] are more sophisticated to analyze and implement.

We consider sensor, communication channel, controller system depicted in Fig. 2, and a continuous nonlinear plant

$$\dot{x} = f(x(t), u(t), w(t)), \tag{7}$$

where x, u, and w are real numbers representing the plant state, control input, and plant disturbance. Furthermore, we assume that for all time $t \geq 0$

$$|w(t)| \leq M. \tag{8}$$

As in (29), the controller constructs the state estimation

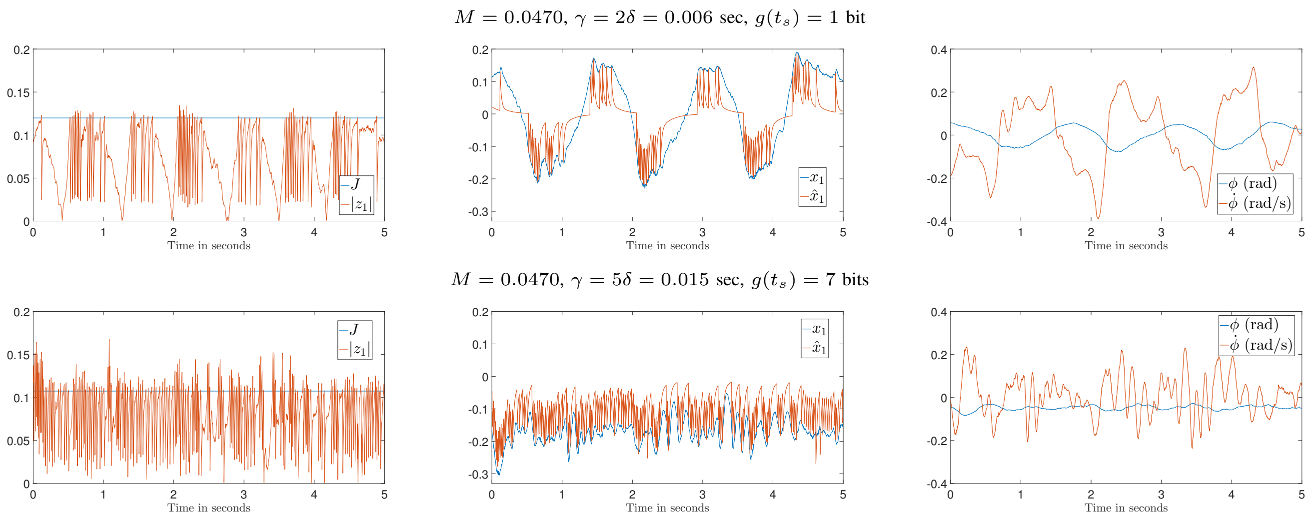

Fig. 6. Experimental results for stabilizing the inverted pendulum over a digital channel with random delay upper bounded by two sampling times (first row) and five sampling times (second row). When $\gamma = 2\delta$, the packet size is 1 bit and when $\gamma = 5\delta$, the packet size becomes 7 bits.

$\hat{x}$, which evolves during the inter-reception times as

$$\dot{\hat{x}} = f(\hat{x}(t), u(t), 0) \quad t \in (t_c^k, t_c^{k+1}), \tag{9}$$

starting at $\hat{x}(t_c^{k+})$ that is constructed by the controller using information received up to time t_c^{k+}. The explicit way to construct $\hat{x}(t_c^{k+})$ will be explained later in this section (see (25)). As discussed in App. B, we assume the sensor can also calculate the controller's state estimate $\hat{x}(t)$.

The state estimation error is defined as (30), thus for $t \in (t_c^k, t_c^{k+1})$ we have

$$\dot{z} = f(x(t), u(t), w(t)) - f(\hat{x}(t), u(t), 0). \tag{10}$$

A triggering occurs at time

$$t_s^k = k(\alpha + \gamma) \tag{11}$$

and the sensor transmits a packet $p(t_s)$ of length $g(t_s)$ to the controller if

$$|z_1(t_s^k)| \geq J, \tag{12}$$

where J and α are non-negative real numbers, γ is the upper bound on the channel delay, $k \in \mathbb{N}$, and $t_s^0 = 0$. We choose $g(t_s)$ such that after decoding we have

$$|z(t_c^{k+})| \leq J. \tag{13}$$

Clearly, the periodic event-triggering scheme (11) and (12) does not exhibit Zeno behavior, meaning that there cannot be infinitely many triggering events in a finite time interval. In fact, we have

$$\Delta'_k = t_s^{k+1} - t_s^k \geq \alpha + \gamma. \tag{14}$$

Assumption 1: The dynamic (7) satisfies the Lipschitz property

$$|f(x, u, w) - f(\hat{x}, u, 0)| \leq L_x|x - \hat{x}| + L_w|w|, \tag{15}$$

where $L_x > 0$, $L_w > 0$, and

$$|z(t)| = |x(t) - \hat{x}(t)| \leq \Upsilon(\gamma). \tag{16}$$

Here for all $0 \leq \vartheta \leq \gamma$, $\Upsilon(\vartheta)$ is defined as follows

$$\Upsilon(\vartheta) := Je^{L_x(\alpha+\gamma+\vartheta)} + \frac{L_w M}{L_x}\left(e^{L_x(\alpha+\gamma+\vartheta)} - 1\right). \tag{17}$$

The reason for choosing the specific value for $\Upsilon(\gamma)$ in (16) will become clear by looking at the following Lemma. If a triggering occurs at time t_s^k, we define

$$\underline{t}^k = \inf\left\{t \in (t_s^{k-1}, t_s^k]\ ;\ |z(t)| = J\right\}. \tag{18}$$

By continuity of z during the inter-reception time, and using (10) and (13), we see that $\underline{t}^k$ is well defined. This definition is used in the next Lemma.

Lemma 1: Consider the plant-sensor-channel-controller model with plant dynamics (7) satisfying Lipschitz property (15), estimator dynamics (9), triggering strategy (11), and (12). Assume $|z(0)| = |x(0) - \hat{x}(0)| < J$ and (13) occurs at all reception times $\{t_c^k\}_{k\in\mathbb{N}}$. Then for all time $t \in [\underline{t}^k, t_c^k)$, where $\vartheta = t - t_s^k$, we have

$$|z(t)| \leq \tag{19}$$

$$\Upsilon_w(\vartheta) := Je^{L_x(\alpha+\gamma+\vartheta)} + \frac{L_w|w|_t}{L_x}\left(e^{L_x(\alpha+\gamma+\vartheta)} - 1\right).$$

Lem. 1 has two important implications. First, if a triggering event does not occur at t_s^k for all $t \in (t_s^{k-1}, t_s^k]$ we have $|z(t)| \leq J$, hence using (13), under the assumptions of Lem. 1 for all time $t \geq 0$ we have

$$|z(t)| \leq \Upsilon_w(\vartheta) \overset{(a)}{\leq} \Upsilon_w(\gamma) \overset{(b)}{\leq} \Upsilon(\gamma), \tag{20}$$

where (a) follows from $\vartheta \leq \gamma$, and (b) follows from (8) and (17). Also, this last inequality explains why we defined the Lipschitz property as (16). The second important implication of Lem. 1 is that for all $k \in \mathbb{N}$ we have $z(t_s^k) \in [-\Upsilon(0), \Upsilon(0)]$.

To construct the packet $p(t_s)$ of length $g(t_s)$, we uniformly quantize the interval $[-\Upsilon(0), \Upsilon(0)]$ into $2^{g(t_s)}$ equal intervals of size $2\gamma(0)/2^{g(t_s)}$. Once the controller receives the packet, it determines the correct sub-interval and selects its center point as the estimate of $z(t_s^k)$, which is represented by $\bar{z}(t_s)$.

In this case, we have

$$|z(t_s) - \bar{z}(t_s)| \leq \Upsilon(0)/2^{g(t_s)}. \tag{21}$$

By (30) we have $x(t_s) = z(t_s) + \hat{x}(t_s)$, thus using $\bar{z}(t_s)$ the controller can construct an estimate of $x(t_s)$ which we denote by $\bar{x}(t_s)$ as follows

$$\bar{x}(t_s) = \bar{z}(t_s) + \hat{x}(t_s). \tag{22}$$

By (21) we deduce that

$$|\bar{x}(t_s) - x(t_s)| \leq \Upsilon(0)/2^{g(t_s)}. \tag{23}$$

For all $t \in [t_s, t_c]$ consider the differential equation

$$\dot{\bar{x}} = f(\bar{x}(t), u(t), 0) \tag{24}$$

with initial condition $\bar{x}(t_s)$ given in (22), and let its solution at time t_c be equal to $\hat{x}(t_c^+)$, namely

$$\hat{x}(t_c^+) = \bar{x}(t_s) + \int_{t_s}^{t_c} f(\bar{x}(t), u(t), 0). \tag{25}$$

We use the above quantization policy to find a sufficient packet size in the next Theorem.

Theorem 1: Consider the plant-sensor-channel-controller model with plant dynamics (7) with Lipschitz property (15), estimator dynamics (9), triggering strategy (11), and (12). Assume $|z(0)| = |x(0) - \hat{x}(0)| < J$, then there exists a quantization policy that achieves (13) for all reception times $\{t_c^k\}_{k\in\mathbb{N}}$ with any packet size

$$g(t_s) \geq \max\left\{0, \log\left(\frac{\Upsilon(0)e^{L_x\gamma}}{J - \frac{L_w M}{L_x}\left(e^{L_x\gamma} - 1\right)}\right)\right\}, \tag{26}$$

provided

$$J \geq \tfrac{L_w M}{L_x}\left(e^{L_x\gamma} - 1\right). \tag{27}$$

In the next assumption we restrict the class of nonlinear systems.

Assumption 2: There exists a control policy $u(t) = \mathfrak{U}(\hat{x}) = \mathfrak{U}(x - z)$ which renders the dynamics (7) ($\dot{x} = f(x, \mathfrak{U}(x-z), w)$) ISS with respect to $z(t)$ and $w(t)$, that is, there exists $\beta' \in \mathcal{KL}$, $\Pi' \in \mathcal{K}_\infty(0)$, and $\psi' \in \mathcal{K}_\infty(0)$ such that for all $t \geq 0$

$$|x(t)| \leq \beta'\left(|x(0)|, t\right) + \Pi'\left(|z|_t\right) + \psi'\left(|w|_t\right).$$

Corollary 1: Under the assumptions of Thm. 1 and Asm. 2 for any packet size lower bounded as (26) there exists a control policy which renders the dynamics (7) ISpS.

Using (14) the triggering rate, the frequency at which triggering occurs, is trivially upper bounded by $(\alpha+\gamma)^{-1}$. As a result, under assumptions of Corollary 1 we deduce that for any information transmission rate (2)

$$R_s \geq \tfrac{1}{\alpha+\gamma}\max\left\{0, \log\left(\tfrac{\Upsilon(0)e^{L_x\gamma}}{J - \frac{L_w M}{L_x}(e^{L_x\gamma}-1)}\right)\right\}, \tag{28}$$

there exists a control law that renders the dynamic (7) ISpS.

The interested reader can find some additional remarks and simulations for the nonlinear systems in App. F and G [19].

VI. Future work

On the theoretical side, future work will explore the theory and implementation of multivariate nonlinear system with uncertainty in its parameters. On the practical validation side, we also plan to test the proposed nonlinear scheme on our inverted pendulum prototype.

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